2.1 Summation of Arithmetic and Geometric Sequences

Summing an Arithmetic Sequence

$$S(n) = \frac{n}{2}(a+l)$$

sum of first n terms of arithmetic series

a: first term, l: last term, n: no of terms

$$S(n) = \frac{n}{2}[2a + (n-1)d]$$

a: first term, n: no. of terms, d: common difference

Example: An arithmetic sequence with the second term 43, and the sixth term 15. Find the sum of the first 25 terms.

Solution:

$T(2) = 43$

$a + (2-1)d = 43 \quad ----(1)$

$T(6) = 15$

$a + (6-1)d = 15 \quad ----(2)$

(2) − (1) becomes

$4d = -28 \quad \therefore d = -7$

$a + (2-1)d = 43 \Rightarrow a = 50$

Sum of first 25 terms

$$S(25) = \underset{n}{\frac{25}{2}}\left[2\underset{a}{(50)} + \underset{n}{(25-1)}\underset{d}{(-7)}\right] = -850$$

Summing a Geometric Sequence

$$S(n) = \frac{a(r^n - 1)}{r-1} \quad \text{where } r \neq 1$$

$$S(n) = \frac{a(1-r^n)}{1-r} \quad \text{where } r \neq 1 \text{ and } r < 1$$

Example: Consider a geometric sequence 7, 14, 28 ---
If the total sum is 1785, how many terms are there?

Solution: $a = 7 \quad r = \frac{14}{7} = 2 \quad S(n) = 1785$

$$S(n) = \frac{a(r^n - 1)}{r - 1} = \frac{7(2^n - 1)}{2 - 1} = 1785$$

$$n = 8 \;//$$

Sum to Infinity

* for $-1 < r < 1$, $r^n \to 0$, as $n \to \infty$

* $S(n) = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$

* $S(\infty) = \frac{a}{1 - r} - \underbrace{\frac{ar^{\infty}}{1 - r}}_{\text{becomes } 0} = \boxed{\frac{a}{1 - r}}$ //

Applications of Summation Formulae

Example 1 Consider 7, 14, 21 ---- Find the sum of all natural numbers between 1 to 100, that are divisible by 7.

Solution: 7, 14, 21 ----

$a = 7 \quad d = 14 - 7 = 7$

the n^{th} term is: $T(n) = 7 + (n-1)7 = 7n$

$7n \leq 100 \quad n \leq 14.3 \quad n = 14$

$$S(14) = \frac{14}{2}[2(7) + (14-1)(7)] = \underline{735}$$

Example 2

Andy opened an account in a bank, and deposit $5000, every month. If the interest rate is 0.5% per month (Compound interest), find the total amount he gets after 1 year.

Solution:

$$T(1) = \$5000\,(1+0.5\%)^{12} = 5000\,(1.005)^{12}$$
$$T(2) = \$5000\,(1+0.5\%)^{11} = 5000\,(1.005)^{11}$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$T(12) = \$5000\,(1+0.5\%)^{1} = 5000\,(1.005)^{1}$$

$a = 5000\,(1.005)^{12}$ $\qquad r = \frac{1}{1.005}$

$$S(n) = \frac{a(r^n-1)}{r-1} \Rightarrow S(n) = \frac{5000\left[\left(\frac{1}{1.005}\right)^{12}-1\right]}{\frac{1}{1.005}-1}$$

$$= \underline{\underline{\$61,986\tfrac{20}{xy}}}$$