MATHS 07 Exponential and Logarithmic Functions

Concept Review

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}} \qquad a^0 = 1$$

$$a^{-n} = \frac{1}{\underbrace{a \times a \times \cdots \times a}_{n \text{ times}}}$$

LAWS OF INDICES

If m and n are integers:

$$\boxed{\begin{aligned} a^m \times a^n &= a^{m+n} \\ a^m \div a^n &= a^{m-n} \quad \text{for } a \neq 0 \\ (a^m)^n &= a^{m \times n} \\ (ab)^m &= a^m b^m \\ \left(\frac{a}{b}\right)^m &= \frac{a^m}{b^m} \quad \text{for } b \neq 0 \end{aligned}}$$

Note: They are also true for rational indices

For example: $x^4 \times (x^2 y^{-1})^3$

$$= x^4 \times (x^2)^3 \times (y^{-1})^3$$

$$= x^4 \times x^6 \times y^{-1 \times 3}$$

$$= x^4 \times x^6 \times y^{-3} = x^{10}/y^3$$

RATIONAL INDICES

Radicals

$2 \times 2 = 4$ also $(-2)\times(-2) = 4$ $\quad \underbrace{2, -2}_{\text{radicals of } 4}$

If $x^2 = y$ $\sqrt{y}$ and $-\sqrt{y}$ are radicals of x

Rational Indices

e.g. if m and n are not integers, the rules still holds

e.g. $3^{\frac{1}{2}} \times 3^{\frac{3}{2}} = 3^{\frac{1}{2}+\frac{3}{2}} = 3^{\frac{4}{2}} = 3^2 = 9$

$$\boxed{\begin{aligned} y^{\frac{1}{n}} &= \sqrt[n]{y} \\ y^{\frac{m}{n}} &= \sqrt[n]{y^m} \\ n, m &> 0 \end{aligned}}$$

example: $8^{2/3} = \left(\sqrt[3]{8}\right)^2$
$= 2^2$
$= 4$ //

So the laws of indices are <u>also</u> true for rational indices. (positive a, b)

SOLVING EQUATIONS INVOLVING INDICES

Method A

if $x^{p/q} = b$ $\quad (q \neq 0)$

then $\left(x^{\cancel{p/q}}\right)^{\cancel{q/p}} = b^{q/p}$

$x = b^{q/p}$ //

Example

$$x^{\frac{1}{3}} = 2$$

$$x^{\frac{1}{3}\times 3} = 2^3$$

$$x = 2^3 = 8$$

Method B

if $a^x = b$ $(a \neq 1)$ exponential equation

tranform into: $m^h = m^k$ ← same base m

$$h = k$$

Example

$$8^{\frac{x}{6}} = 16$$

$$(2^3)^{\frac{x}{6}} = 2^4$$

$$2^{\frac{x}{2}} = 2^4$$

express into same base "2"

therefore $\frac{x}{2} = 4$ $x = 8$

LOGARITHMS (LOG)

$$\boxed{\begin{array}{l}\text{Log to the base 10} \\ \text{If } n = 10^x \text{, then } x = \log n\end{array}}$$

e.g. $\log_{10}(100) = 2$ or $\log(100) = 2$

$\log_{10}(0.01) = -2$ or $\log(0.01) = -2$

Properties of Logarithms

1. For $\log a = b$ then $a = 10^b$

   example : $\log \underline{100} = \underline{2}$ $\quad \underline{100} = 10^{\underline{2}}$

2. For $\log 10^b = b$

   Then $\log 1 = \log 10^0 = \underline{0}$

   $\log 10 = \log 10^1 = \underline{1}$

3. $\log (MN) = \log M + \log N$ $\left(\begin{array}{c}\text{not equal to} \\ \log M \times \log N\end{array}\right)$

4. $\log M/N = \log M - \log N$ $\left(\begin{array}{c}\text{not equal to} \\ \log M / \log N\end{array}\right)$

5. $\log M^n = n \log M$

Example

find $\log 8 / \log 2 = \log 2^3 / \log 2$

$$= \frac{3 \cancel{\log 2}}{\cancel{\log 2}} = 3$$

Use Log to solve equations

Method A change to same base, log it first.

example 1: find x $2^x = 8 \Rightarrow 2^x = 2^3$

$x = 3$ //

example 2: find x $2^x = 7 \Rightarrow x\log 2 = \log 7$

$x = \dfrac{\log 7}{\log 2} = 2.81\ldots$ //

Method B solving equations with log functions

example 1: $\log x = 2 \Rightarrow x = 10^2 = 100$ //

example 2: $\log (x+1)^3 = 3 \Rightarrow 3\log(x+1) = 3$

$\Rightarrow \log(x+1) = 1$

$\Rightarrow x+1 = 10^1$ $x = 9$ //

example 3: $\log(x+6) + \log(x-3) = 1$

$\log[(x+6)(x-3)] = 1$

$(x+6)(x-3) = 10$

$x^2 + 3x - 18 = 10$

$x^2 + 3x - 28 = 0$

$(x+7)(x-4) = 0$ $x = 4$ or -7

negative log number rejected

LOGARITHMS TO THE BASE a

$$\boxed{\begin{array}{l} \text{If } n = a^x, \text{ where } a > 0, \; a \neq 1 \\ \text{Then } x = \log_a n \end{array}}$$

example

1. $5^2 = 25$ $\quad \log_5 25 = 2$, base 5
2. $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ $\quad \log_{\frac{1}{2}}\left(\frac{1}{16}\right) = 4$, base $\frac{1}{2}$

Log to base a has the same properties as $\log_{10}$

1. $\log_a a^b = b$
2. $\log_a a = 1$
3. $\log_a 1 = 0$
4. $\log_a (MN) = \log_a M + \log_a N$
5. $\log_a \frac{M}{N} = \log_a M - \log_a N$
6. $\log_a M^n = n \log_a M$

If $a = 10$, then, it is $\log_{10}$ base

Change of Base Formula

$$\boxed{\log_a M = \frac{\log_b M}{\log_b a}}$$

for any positive no. M, a, b where $a, b \neq 1$

Applications of Log

Loudness of Sound : $L = 10 \log_{10} \frac{I}{I_0}$

I : intensity of sound in (W/m^2)
I_0 : min audible sound $10^{-12}\ W/m^2$

Ex/ At normal loudness of 60dB $60 = 10 \log \frac{I}{10^{-12}}$

$$I = 10^{-6}\ W/m^2$$

Earthquake (Richter Scale)

E: Energy released in (J) $\log_{10} E = 4.8 + 1.5M$

Richter Scale

Ex/ If earthquake release 1.122×10^{16} J energy
The Richter Scale magnitude (M) is :

$$\log(1.122 \times 10^{16}) = 4.8 + 1.5M$$

$$M = 7.5$$

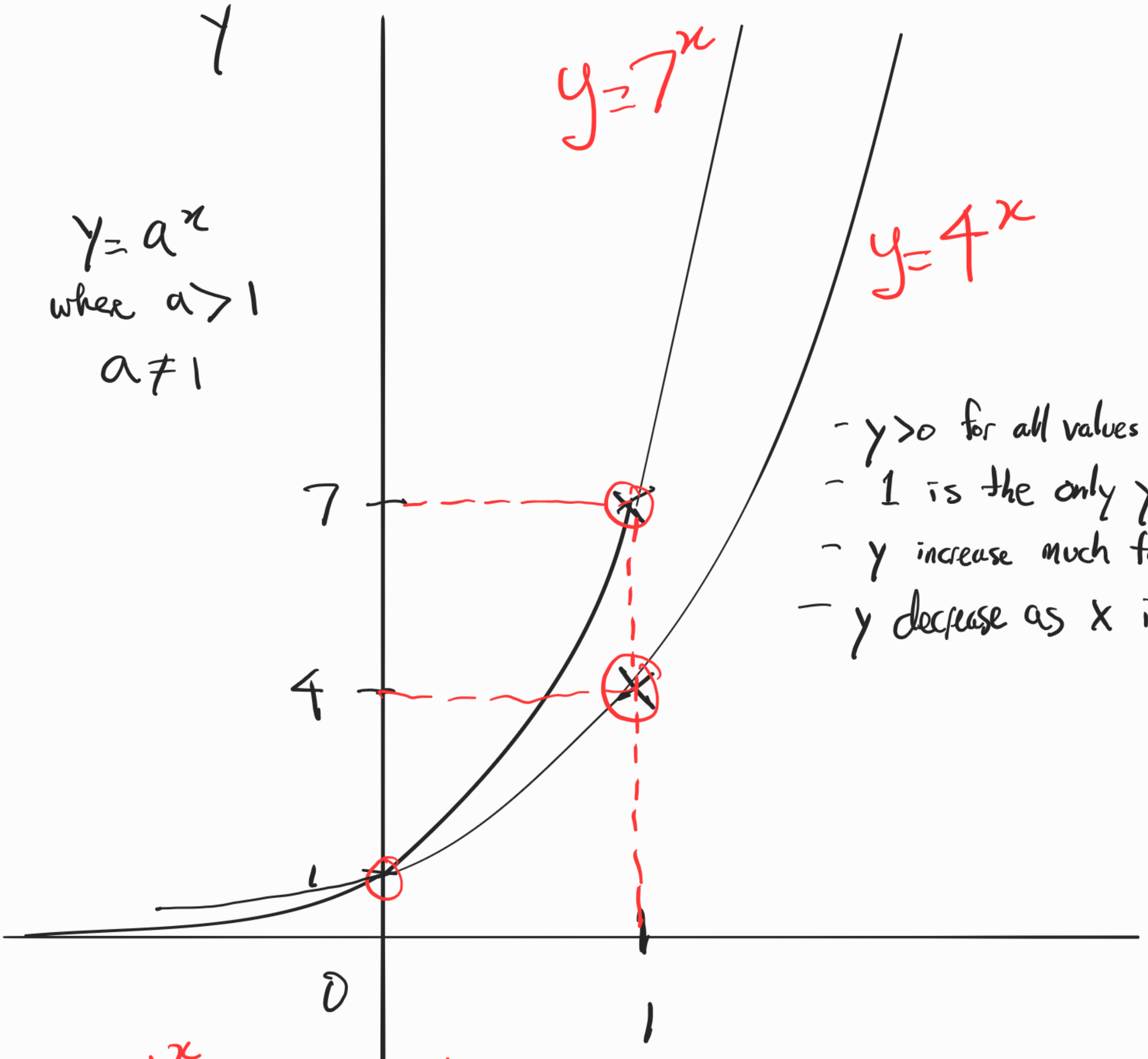
Exponential and Logarithmic Functions

Plot $y = a^x$, for $a = 4, 7$

See next page

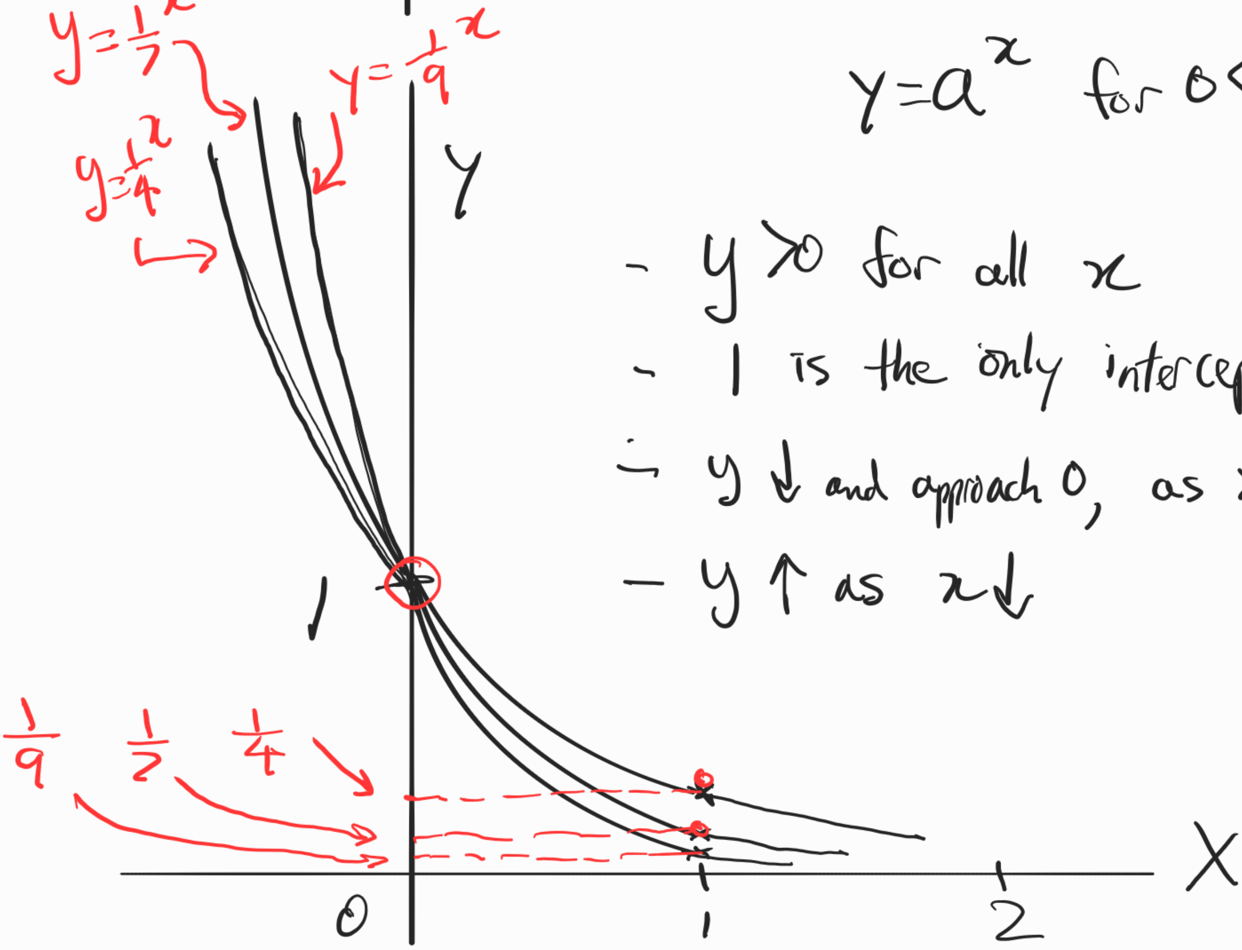
$y = a^x$ where $a > 1$

$a \neq 1$



- $y > 0$ for all values of x
- 1 is the only y intercept
- y increase much faster than x
- y decrease as x increase

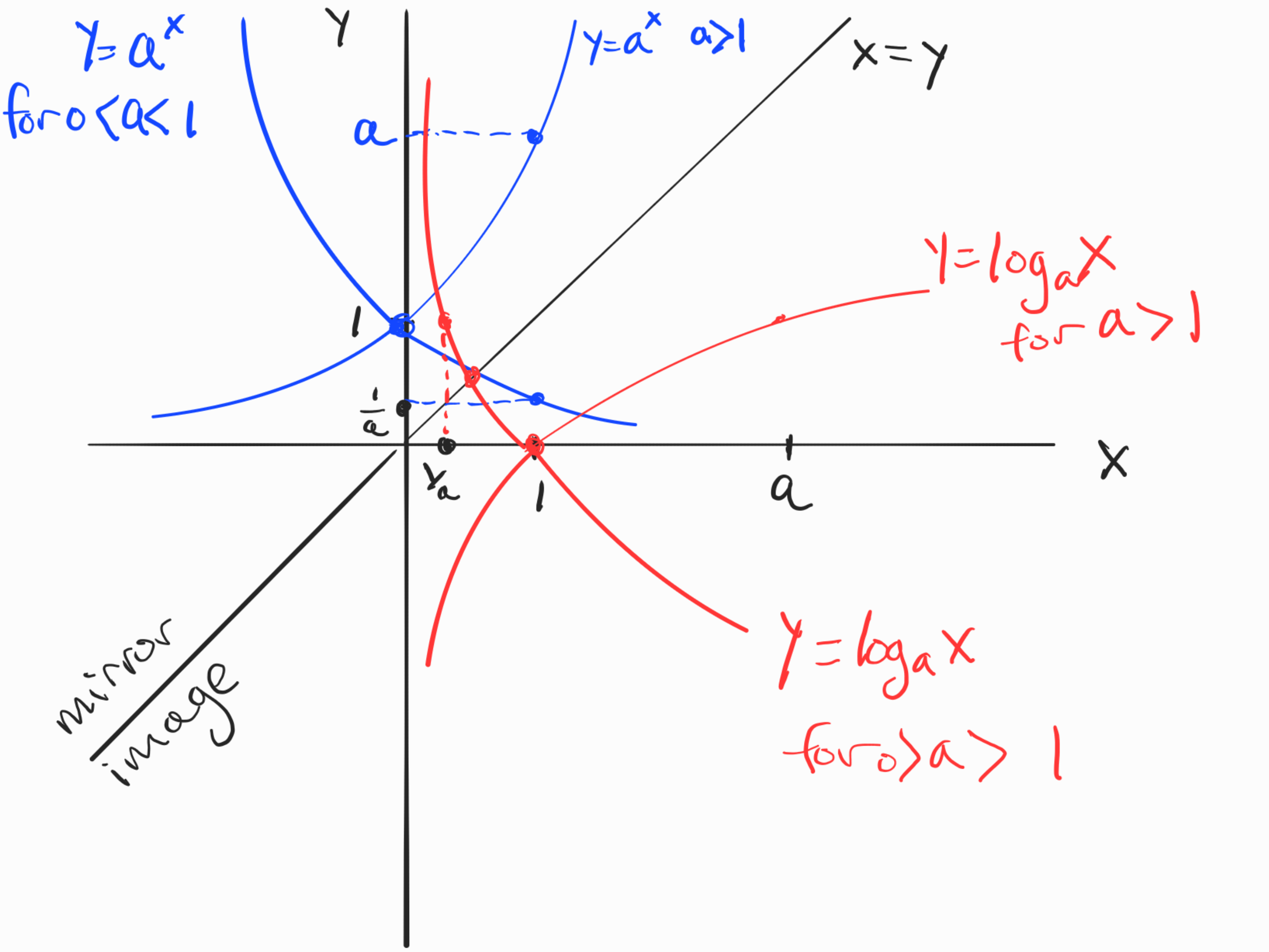
$y = a^x$ for $0 < a < 1$



- $y > 0$ for all x
- 1 is the only intercept
- $y \downarrow$ and approach 0, as $x \uparrow$
- $y \uparrow$ as $x \downarrow$

Logarithmic Functions and their graphs

(just swap the x, y axis graphs from above)



$$y = \log_a x \quad \underline{\text{vs}} \quad y = a^x$$

- They are inverse functions
- Swap x and y axis will inverse the function
- Mirror image over the line $x = y$

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