Maths 06 - More about Polynomials

Polynomials

general : $a_n x^n + a_{n-1} x^{n-1} + \dots a_2 x^2 + a_1 x + a_0$

* x is a variable , n is the order
* a_n, a_{n-1} , a_{n-2} ----- a_0 are number
* $a_n \neq 0$ and a_0 = constant

Ex : $P(x) = \underset{a_3}{4} x^3 - \underset{a_2}{2} x^2 + \underset{a_1}{3} x + \underset{a_0}{1}$

3rd degree polynomial
$n = 3$

Addition and Subtraction

* group like terms together
* Perform (+) or (−) with same order terms

ADD
e.g 1: $(3x^3 + 2x^2 - 4x + 5) + (x^3 - 4x^2 + 3x + 1)$

$= 3x^3 + 2x^2 - 4x + 5 + x^3 - 4x^2 + 3x + 1$

$= \underline{3x^3 + x^3} + \underline{2x^2 - 4x^2} \underline{- 4x + 3x} + \underline{5 + 1}$

$= 4x^3 - 2x^2 - x + 6$ //

Alternatively:

$$\begin{array}{rrrrr} & 3x^3 & +\,2x^2 & -\,4x & +\,5 \\ +) & x^3 & -\,4x^2 & +\,3x & +\,1 \\ \hline & 4x^3 & -\,2x^2 & -\,x & +\,6 \end{array}$$

SUBTRACT

e.g. 2: $(5x^3 - 3x^2 + 4x - 2) - (4x^3 + 5x - 6)$

$= 5x^3 - 3x^2 + 4x - 2 - 4x^3 - 5x + 6$

$= \underline{5x^3 - 4x^3} - 3x^2 + \underline{4x - 5x} \underline{- 2 + 6}$

$= x^3 - 3x^2 - x + 4$

Alternatively

$$\begin{array}{rrrrr} & 5x^3 & -\,3x^2 & +\,4x & -\,2 \\ -) & 4x^3 & +\,0x^2 & +\,5x & -\,6 \\ \hline & x^3 & -\,3x^2 & -\,x & +\,4 \end{array}$$

Multiplication

distributive law $a(b+c) = ab + bc$

$(a+b)(c+d) = ac + ad + bc + bd$

Multiply

e.g.: $(2x^3 - x^2 + 4x + 3)(2x + 3)$

$= (2x^3 - x^2 + 4x + 3)2x + (2x^3 - x^2 + 4x + 3)3$

$= 4x^4 - 2x^3 + 8x^2 + 6x + 6x^3 - 3x^2 + 12x + 9$

$= 4x^4 + 4x^3 + 5x^2 + 18x + 9$

Alternatively

$$\begin{array}{r} 2x^3 - x^2 + 4x + 3 \\ \times) \qquad\qquad 2x + 3 \\ \hline 4x^4 - 2x^3 + 8x^2 + 6x \qquad \\ +) \quad 6x^3 - 3x^2 + 12x + 9 \\ \hline 4x^4 + 4x^3 + 5x^2 + 18x + 9 \end{array}$$

Division

① cancel factor, if completly divisible

e.g. $(20x^2 + 15x) \div (5x)$

$$= \frac{20x^2 + 15x}{5x} = \frac{(4x+3)\cancel{5x}}{\cancel{5x}} = 4x + 3$$

dividend

divisor

Quotient

② Use long division, for complex operation

e.g. 1

$$\begin{array}{r} x^2 + 5x + 5 \\ x-2 \overline{) x^3 + 3x^2 - 5x + 6} \\ \underline{x^3 - 2x^2 \qquad\qquad} \\ 5x^2 - 5x \qquad \\ \underline{5x^2 - 10x \qquad} \\ 5x + 6 \\ \underline{5x - 10} \\ 16 \end{array}$$

quotient

dividend

divisor (degree 1)

remainder (degree 0)

Dividend = Divisor × Quotient + Remainder

Remainder Theorem

To find the remainder without doing the division

when polynomial $P(x)$ is divided by $(mx-n)$

remainder $R = P(\frac{n}{m})$

e.g. $x^3 + x^2 - 3x + k$ divided by $(x-2)$

The remainder is -4, find k

$P(x) = x^3 + x^2 - 3x + k$

$P(2) = 2^3 + 2^2 - 3(2) + k = -4$

$= 6 + k = -4$

$\underline{\underline{k = -10}}$

$mx - n$

$m=1 \quad n=2$

$P(\frac{n}{m}) = P(\frac{2}{1})$

$= P(2)$

Factor Theorem (i.e. remainder = 0)

To find out if the polynomial can be <u>fully divided</u> by a factor

For $P(x)$, if $P(\frac{n}{m}) = 0$, then $(mx-n)$ is a factor

or

If $(mx-n)$ is a factor of $P(x)$, then $P(\frac{n}{m}) = 0$

e.g.: $P(x) = x^3 - 4x^2 + x + 6$

which is a factor? $(x+1)$ or $(2x-1)$

Try $(x+1)$:

for $(mx-n) \Rightarrow (x+1) \Rightarrow m=1 \quad n=-1 \quad P\left(\frac{n}{m}\right) = P(-1)$

$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$ //

It is a factor of $P(x)$

Try $(2x-1)$:

for $(mx-n) \Rightarrow m=2 \quad n=1 \Rightarrow P\left(\frac{n}{m}\right) = P\left(\frac{1}{2}\right)$

$P\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 6 = \frac{45}{8} \neq 0$

It is NOT a factor

Find a factor of k by factor theorem

e.g. $P(x) = 2x^3 + 3x^2 - 2x - 3$ factor: $(mx-n)$ $P\left(\frac{n}{m}\right)$

$m = 2 \times 1$ $\qquad$ $n = \pm 1 \times \pm 3$

$x < \begin{matrix} \pm 1 \to x \pm 1 \\ \pm 3 \to x \pm 3 \end{matrix}$ $\qquad$ $2x < \begin{matrix} \pm 1 \to 2x \pm 1 \\ \pm 3 \to 2x \pm 3 \end{matrix}$

8 possible factors, try them out

$P(1) = 2(1)^3 + 3(1)^2 - 2(1) - 3 = 0$ Success!

Therefore $(x-1)$ is a factor

long division

$\therefore P(x) = 2x^3 + 3x^2 - 2x - 3 = (x-1)(2x^2 + 5x + 3)$

$= (x-1)\underbrace{(2x+3)(x+1)}_{\text{by factorization}}$

The GCD and LCM of $P(x)$

GCD ↳ greatest common divisor (similar to HCF)

LCM ↳ least common multiple (similar to LCM)

Example:

find the GCD and LCM of: $x^2-7x+12$
x^2-6x+9
$2x^2-11x+15$

$x^2-7x+12$: $(x-3)(x-4)$
x^2-6x+9 : $(x-3)(x-3)$
$2x^2-11x+15$: $(x-3)(2x-5)$

GCD : Look common factors

ANSWER : $(x-3)$ //

LCM : Look for multiple combination of all 3 $P(x)$

Answer : $(x-3)(x-3)(x-4)(2x-5)$ //

Factorization and Simplification

e.g 1: Simplify : $\dfrac{x^2-1}{3x+3} \times \dfrac{9}{x^2+3x-4}$

$x^2-1=(x+1)(x-1)$
$3x+3=3(x+1)$
$x^2+3x-4=(x-1)(x+4)$

$$= \frac{\cancel{(x+1)}\cancel{(x-1)}}{{}_1\cancel{3}\cancel{(x+1)}} \times \frac{\cancel{9}\,3}{\cancel{(x-1)}(x+4)}$$

$$= \frac{3}{x+4}$$ //

e.g. 2:

$$\frac{x+2}{x+1} \div \frac{x+2}{x+3} - 1$$

$$= \frac{\cancel{x+2}}{x+1} \times \frac{x+3}{\cancel{x+2}} - 1$$

$$= \frac{x+3}{x+1} - 1$$

$$= \frac{x+3}{x+1} - \frac{x+1}{x+1}$$

$$= \frac{(x+3)-(x+1)}{x+1} = \frac{2}{x+1}$$

END 完